FUZZY RELATIONAL OPTIMIZATION AND ITS APPLICATIONS

Synopsis

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1. INTRODUCTION

The fuzzy set theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modeling language, well suited for situations in which fuzzy relations, criteria and phenomena exist. This theory grew slowly in the 1960s and 1970s of the last century. In the second half of the 1970s, the first successful practical application, in the control of technological processes through fuzzy rule-based systems, was heating system.

In classical set theory, we deal with two value logic in which two possibilities arise for the existence of an element in a classical set. Either an element belongs to the set or does not belong to the set. A classical set A over a universal set X is usually described by a function known as the characteristic function $\chi_A : X \to \{0,1\}$, as follows:

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$$

When we talk about classical logic, the truth value of any classical proposition is either 0 or 1. If the proposition is true, then its truth value is 1 and if the proposition is false then its truth value is 0. For example, if we consider two propositions: P_1 : Taj Mahal is in Agra, P_2 : 2 is less than 1. Then the truth value of P_1 is 1 and P_2 is 0. But in fuzzy theory we deal with the data that consists of impossible measurements of essential parameters of some processes, as the temperature inside some melting charge of glass or the homogeneity of a mixture inside some tank vessel etc. It may be that there does not exist a measurement scale at all, as for the evaluation of offensive smells, as for medical diagnoses by touching or as for the evaluation of the taste of foods. In fuzzy set theory, we deal with the data that partially belongs to the set or the data which consists of indefinite information. So to define the existence of this data, Zadeh [76] introduced the concept of membership function.

If X is a collection of objects denoted generically by x, then a fuzzy set A on X is a set of ordered pairs: $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A(x)$ is called the membership function (generalized characteristic function) which maps X to the membership space. Its range is [0,1].

A classical relation R on sets X and Y is a classical set defined by the characteristic function $R: X \times Y \to \{0,1\}$ consisting the collection of pairs (x, y) where $x \in X, y \in Y$ and xRy. On the other hand, a fuzzy relation R on sets X and Y is a fuzzy set on $X \times Y$ defined by the membership function $R: X \times Y \to [0,1]$ where the degree of association or membership grade is also available as an additional information. In short, a fuzzy set can be viewed as an extension of a classical set and the concept of a fuzzy relation can be viewed as an extended form of a classical relation.

An application of fuzzy logic can be formulated in terms of solving fuzzy relation equations (FRE). FRE plays an important role in fuzzy set theory. Several applications of FRE can be found in [8,44,48,49,51]. To understand FRE, we first understand the composition of fuzzy relations.

Here, we discuss two important compositions. In mathematical terms if P and Q are fuzzy relations from set X to Y and set Y to Z respectively, then the sup-*t* composition of P and Q is a fuzzy relation denoted by $P \overset{t}{\circ} Q$ from set X to Z defined as

$$[P \overset{t}{\circ} Q](x, z) = \sup_{y \in Y} t\{P(x, y), Q(y, z)\}, \forall x \in X, \forall z \in Z$$

and inf-ω_t composition of P and Q is a fuzzy relation denoted by $P \overset{\omega_t}{\circ} Q$ from X to Z defined as

$$[P \overset{\omega_t}{\circ} Q](x, z) = \inf_{y \in Y} \omega_t\{P(x, y), Q(y, z)\}, \forall x \in X, \forall z \in Z$$

where $\omega_t(x, y) = \sup\{a \in [0,1] \mid t(a, x) \le y\}$ and t denotes a continuous t-norm. Zadeh [76] introduced the most common max-min composition which is used when the value of one object does not affect the other.

2. FUZZY RELATION EQUATIONS

The first study of fuzzy relation equations can be found in Sanchez [50] using max-min composition. Generally, fuzzy set theory have a number of properties that make it suitable for formalizing the uncertain information upon which many applied concepts such as medical diagnosis and treatment are usually based. Because of this fact, the theory of FRE was originally applied in problems of the medical diagnosis [48,51]. Nowadays, it is well-known that many issues associated with body knowledge can be treated as FRE problems. FRE theory constitutes a good tool to model process in

knowledge engineering [8]. This theory has been applied in many fields including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. FRE are becoming extremely important in order to investigate the optimal solution of the inverse problem through restrictive conditions for the availability of the solution of such inverse problems. Various real world problems such as networking, medical etc., can be formulated as optimization problems constrained with FRE.

The resolution of FRE is a dominant problem in fuzzy set theory. It is an *NP*-hard problem. So, there is no exact method to solve and obtain the complete solution set of FRE. Thus finding a particular method or polynomial time algorithm to solve FRE, is an important topic of research.

The mathematical formulation of FRE is defined as follows:

Let X , A and B be the fuzzy binary relations from P to Q, Q to R and P to R of order $s \times m$, $m \times n$ and $s \times n$ respectively. Let I , J and K be the index sets given as $I = \{1, 2, \ldots, m\}$, $J = \{1, 2, \ldots, n\}$ and $K = \{1, 2, \ldots, s\}$. Assume that the three relations X, A and B are constrained in such a way that

$$X \circ A = B \tag{1}$$

where $\circ$ denotes the max-min composition of X and A. Equation (1) can be further written as

$$\max_{i \in I} \min (x_{ki}, a_{ij}) = b_{kj}, \forall\, k \in K, \forall j \in J \tag{2}$$

Equation (1) consists of $s \times n$ simultaneous FRE of the form (2). If two components in equation (1) are given and one is unknown, then the equation is known as FRE. If X and A are given, then the problem is trivial and the system is always consistent with a unique solution. The problem becomes complicated when either X or A is unknown, because in this case, the system may be inconsistent i.e. the system may have no solution and if it has a solution then it may not be unique. We consider only the case when X is unknown and A and B are known. Because if we have a method for solving this problem, then the other problem (when A is unknown and X and B are known) can be solved employing inverse of the relational matrices, and rewriting the system as

$$A^{-1} \circ X^{-1} = B^{-1} \tag{3}$$

To solve the system (1), it is easy to solve each equation separately, i.e. the system (1) can be decomposed into system of equations of the form

$$x_k \circ A = b_k \tag{4}$$

for all $k \in K$ where $x_k = [x_{ki} \mid i \in I]$ and $b_k = [b_{kj} \mid j \in J]$. Since (4) is simpler form of (1), we need only method for solving equations of the form (4), in order to arrive at a solution. Moreover, the system of FRE (4) can be further simplified as

$$x \circ A = b \tag{5}$$

where $x = [x_i \mid i \in I], A = [a_{ij} \mid i \in I, j \in J]$ and $b = [b_j \mid j \in J]$. The resolution problem of FRE is to find a solution vector $x = [x_i \mid i \in I]$, $x_i \in [0,1]$, $\forall i \in I$ such that (5) holds. To obtain the solution set of FRE, first the consistency of the system of FRE is checked. A consistent system of FRE can be characterized by a unique maximum solution and finite minimal solutions or by a unique minimum solution and finite maximal solutions. In the next sub-section we discuss the general forms of different types of FRE.

2.1. System of sup-$\otimes$ fuzzy relation equations

Let $A = [a_{ij}]$ be a matrix of dimension $m \times n$ and $b = [b_j]$ be an n dimensional vector with $a_{ij} \in [0,1]$, $b_j \in [0,1]$ and $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ be the index sets. The system of FRE with sup-$\otimes$ composition are defined as,

$$x \circ A = b \tag{6}$$

where $\circ$ denotes the sup-$\otimes$ composition of x and A, $\otimes$ being a continuous t-norm, i.e.

$$\max_{i \in I}(x_i \otimes a_{ij}) = b_j, \forall j \in J \tag{7}$$

The resolution problem of FRE (6) is to determine a vector $x = [x_i \mid i \in I]$, $x_i \in [0,1], \forall i \in I$ such that (7) holds. Let the solution set of the system is given as $X(A,b) = [x \in [0,1]^m \mid x \circ A = b]$. If $X(A,b) \neq \phi$, then the system is called consistent, otherwise it is called inconsistent. For an $x^1, x^2 \in X$, we say $x^1 \leq x^2$ if $x_i^1 \leq x_i^2, \forall i \in I$. Then '$\leq$' forms a partial ordering relation on X and

$(X,\leq)$ becomes a lattice. $\hat{x} \in X(A,b)$ is said to be the maximum solution, if $x \leq \hat{x}, \forall\, x \in X(A,b)$. Similarly, $\check{x} \in X(A,b)$ is said to be a minimal solution, if $x \leq \check{x}$ implies $x = \check{x}$, $\forall\, x \in X(A,b)$. If $X(A,b) \neq \phi$, then, in general it is a non-convex set which can be completely determined by unique maximum solution $\hat{x}$ and several minimal solutions $\check{x}$.

The maximum solution is obtained by

$$\hat{x} = A \to b = [\min_{j \in J} (a_{ij} \to b_j)]_{i \in I} \tag{8}$$

where $a_{ij} \to b_j = \sup\{x_i \in [0,1] \mid (x_i \otimes a_{ij}) \leq b\}$. The complete solution set of the system of FRE (7) is given as

$$X(A,b) = \bigcup_{\check{x} \in \check{X}(A,b)} \{x \in [0,1]^m \mid \check{x} \leq x \leq \hat{x}\} \tag{9}$$

where $\check{X}(A,b)$ denotes the set of all minimal solutions of FRE (7).

2.2. System of interval valued sup-$\otimes$ fuzzy relation equations

Let $A^- = [a_{ij}^-]$, $A^+ = [a_{ij}^+]$ be matrices of dimension $m \times n$ with $0 \leq a_{ij}^- \leq a_{ij}^+ \leq 1$, and $b^- = [b_j^-]$, $b^+ = [b_j^+]$ be *n* dimensional vectors such that $0 \leq b_j^- \leq b_j^+ \leq 1$. The system of interval-valued FRE with sup-$\otimes$ composition is defined as

$$b_j^- \leq \max_{i \in I} (x_i \otimes a_{ij}) \leq b_j^+, \forall a_{ij} \in [a_{ij}^-, a_{ij}^+] \tag{10}$$

where $\otimes$ denotes a continuous t-norm. When $A^- = A^+$ and $b^- = b^+$ in the equation (10), then it converted to (7). So we can say that the sup-$\otimes$ FRE is a special case of interval-valued FRE. The solution set of interval-valued FRE can be completely determined by one maximum solution and a finite number of minimal solutions. The maximum solution of interval-valued FRE (10) can be obtained by assigning

$$\hat{x} = A \to b = [\min_{j \in J} (a_{ij}^+ \to b_j^+)_{i \in I}]_{i \in I} \tag{11}$$

where $a_{ij}^+ \to b_j^+ = \sup\{x_i \in [0,1] \mid (x_i \otimes a_{ij}^+) \leq b_j^+\}$.

2.3. System of inf-→ fuzzy relation equations

Let $A=[a_{ij}]$ be a matrix of dimension $m\times n$ and $b=[b_j]$ be an *n* dimensional vector with $a_{ij}\in[0,1]$, $b_j\in[0,1]$ and $I=\{1,2,\ldots,m\}$ and $J=\{1,2,\ldots,n\}$ be the index sets. The system of inf-$\rightarrow$ FRE is defined as

$$x\circ^{\rightarrow}A=b \tag{12}$$

where $\circ^{\rightarrow}$ denotes the inf-$\rightarrow$ composition of x and A, $\rightarrow$ being the residual implicator, defined as

$$\inf_{i\in I}(x_i\rightarrow a_{ij})=b_j,\forall j\in J \tag{13}$$

Given A and b, the resolution of (13) means to find a vector $x=[x_i\mid i\in I]$, $x_i\in[0,1],\forall i\in I$ such that (13) holds. The maximum solution of inf-$\rightarrow$ FRE (13) can be obtain by assigning

$$\hat{x}=b\circ^{\rightarrow}A^{-1} \tag{14}$$

If x and b are given, then the resolution of FRE (12) means to find matrix $A=[a_{ij}]$ such that (13) holds. The minimum solution of (12) is given by $A=x^{-1}\circ b$. As there is no particular method to determine all the minimal solutions for the system of sup-$\otimes$ FRE, similarly, there is no well-known method by which we can determine all maximal solutions of inf-→ FRE.

2.4. Bipolar fuzzy relation equations

Let $A^{+}=[a_{ij}^{+}]$ and $A^{-}=[a_{ij}^{-}]$ be matrices of dimension $m\times n$ and $b=[b_j]$ is an *n* dimensional vector with $a_{ij}^{-}\in[0,1]$, $a_{ij}^{+}\in[0,1]$ and $b_j\in[0,1]$. The system of bipolar FRE with sup-$\otimes$ composition is defined as

$$x\circ A^{+}\vee\overline{x}\circ A^{-}=b \tag{15}$$

where $\circ$ denotes the sup-$\otimes$ composition and $\otimes$ denotes a continuous t-norm. Equation (15) can also be further written as

$$\max_{i\in I}[\max(x_i\otimes a_{ij}^{+}),\max(\overline{x}_i\otimes a_{ij}^{-})]=b_j,\forall j\in J \tag{16}$$

where $I=\{1,2,...,m\}$ and $J=\{1,2,...,n\}$ are the index sets. The resolution problem of FRE (15) is to determine a vector $x=[x_i \mid i\in I],\ x_i\in[0,1], \forall i\in I$ such that (16) holds. The solution set of bipolar FRE whenever nonempty, can be characterized by a finite set of maximal and minimal solution pairs.

2.5. Fuzzy relation inequalities

Let $A=[a_{ij}]$ and $A^{'}=[a_{ik}^{'}]$ be the fuzzy matrices of dimension $m\times n$ and $m\times s$ respectively, with $a_{ij}\in[0,1]$, $a_{ik}^{'}\in[0,1]$ and $b=[b_j]$, $b^{'}=[b_k]$ be the fuzzy vectors of dimension n and s respectively, with $b_j\in[0,1]$, $b_k^{'}\in[0,1]$. The system of fuzzy relation inequalities are defined as

$$x\circ A\geq b\wedge x\circ A^{'}\leq b^{'} \tag{17}$$

where '$\circ$' denotes the sup-$\otimes$ composition and $\otimes$ denotes a continuous t-norm. Equation (17) can be further written as

$$\max_{i\in I}(x_i\otimes a_{ij})\geq b_j, \forall j\in J \text{ and } \max_{i\in I}(x_i\otimes a_{ik}^{'})\leq b_k^{'}, \forall k\in K \tag{18}$$

where $I=\{1,\ 2,\dots,m\}$, $J=\{1,\ 2,\dots.,n\}$ and $K=\{1,\ 2,\dots.,s\}$ are the index sets. The resolution problem of (17) means to find a vector $x=[x_i \mid i\in I],\ x_i\in[0,1], \forall i\in I$ such that (18) holds. The complete solution set can be determined by one maximum solution and finite number of minimal solutions.

3. FUZZY OPTIMIZATION PROBLEMS

The general model of optimization problems subject to the system of sup-$\otimes$ FRE, $\otimes$ being a continuous t-norm is expressed as:

$$\begin{aligned}&\text{Optimize } \{Z_k(x)\}_{k\in K},\ K=\{1,2,\dots,s\}\\ &\text{s.t. } x\circ A=b\\ &x_i\in[0,1], \forall i\in I\end{aligned}$$

where $A=[a_{ij}]$ is a matrix of dimension $m\times n$ and $b=[b_j]$ is an n dimensional vector with $a_{ij}\in[0,1]$, $b_j\in[0,1]$, $I=\{1,2,\dots,m\}$ is the index set and $\circ$ denotes the sup-$\otimes$ composition of x and A, $\otimes$ being a continuous t-norm. If $|K|=1$ and $Z_k(x)$ is a linear function, then problem is a

linear optimization problem constrained with sup-$\otimes$ FRE. If $|K|=1$ and $Z_k(x)$ is a nonlinear function, then problem a nonlinear optimization problem constrained with sup-$\otimes$ FRE. For $|K|>1$, it is a multi-objective optimization problem, where $Z_k(x)$ is a linear function or a nonlinear function. Similarly, other variants of fuzzy optimization problems can be defined taking different types of FRE or FRI.

4. LITERATURE REVIEW

FRE have played an important role in fuzzy modeling [6]. Many researchers have worked on FRE. There are several types of equations which we have discussed in previous section, each equation consisting of a specific composition of fuzzy relations. The first paper on FRE with max-min composition was published by Sanchez [50]. Sanchez [51] gave an application of FRE in the diagnosis problem, as it contains information like medical descriptions and assertions which are generally ambiguous, uncertain and incomplete information. Later on, Di Nola et al. [7] solved the sup-$\otimes$FRE, where$\otimes$is an Archimedean t-norm, and characterized the solution set by one maximum solution and finitely many minimal solutions. Stamou and Tzafestas [53] considered the FRE with sup-$\otimes$ composition, $\otimes$ being an Archimedean t-norm and discussed the solvability criteria for FRE for any t-norm composition. Luoh et al. [36] proposed a matrix-pattern based computer algorithm to solve max-min and max-product FRE.

Markovskii [37] considered max-product FRE and showed that solving these equations is closely related with the covering problem, which belongs to the category of *NP*-hard problems. The concepts of essential, non-essential, semi-essential and super-essential variables were suggested. It was also shown that the relation between max-min FRE and the covering problem is more complex and it is no longer possible to establish a one-to-one mapping of minimal solutions and the solutions of some covering problem.

Li and Fang [26] investigated the resolution of a finite system of FRE with sup-*t* composition, *t* being a continuous t-norm. They divided the system of equations into two categories depending on the involved t-norm. When the norm is Archimedean, the minimal solutions correspond one-to-one to the irredundant coverings of a set covering problem. In case of non-Archimedean, it only corresponds to a

subset of constrained irredundant coverings of a set covering problem. Shieh [52] proposed an efficient algorithm to show one-to-one relationship between minimal coverings of the covering problem and minimal solutions of the reduced max-product FRE. Lin et al. [32] generalized the covering method for FRE with max-continous *t*-norm. For solving such problem, they proposed two approaches; first transforming FRE into a covering problem and the second transforms the covering problem into a system of max-product FRE. Molai [41] discussed the concept of lower triangular matrix L and upper triangular matrix U for max-product FRE. They expressed fuzzy square matrix A in terms of L and U such that $A = L \circ U$, where '$\circ$' denotes the max-product composition. They presented an algorithm find the matrices L and U. The algorithm proposed by them finds the solution set without finding its minimal solutions and the maximum solution.

As we have in above discussion, all the researchers gave us methods for solving FRE with different compositions. But the problem becomes different when we associate it with the objective function. Optimization model with a linear objective function subject to the system of max-min FRE was first considered by Fang and Li [10]. They designed a procedure for solving the problem by converting it to an equivalent 0-1 integer programming problem and solved it using the well-known branch and bound method. Loetamonphong and Fang [33] considered similar problem subject to max-product FRE. They defined two sub-problems by separating the negative and non-negative coefficients of the objective function, and then obtained the optimal solution by combining the optimum solution of both sub-problems. The sub-problem with negative coefficients was easily optimized by the maximum solution of the set of solutions while the other sub-problem was converted into a 0-1 integer programming problem and solved by branch and bound method. Guu et al. [65] provided a necessary condition for the linear optimization problem subject to max-min FRE. This necessary condition provides that, for a minimal solution, each of its components is either 0 or attains the value of the maximum solution. From the algorithmic viewpoint, it enables to pre assign values for some variables and helps in reducing the problem size. Pandey and Srivastava [43] solved the similar problem subject to max-product FRE, by converting it into 0-1 integer programming problem and applied branch and bound method with forward jumping-tracking technique. Pandey [42] studied similar problem subject to sup-

T FRE, T being a continuous t-norm. Using the concept of binding, non-binding, basic, non-basic variables, they proposed an algorithm for the given problem.

Wu and Guu [64] considered a linear optimization problem subject to max-min FRE. They added new theoretical results and rules and established necessary conditions possessed by the optimal solution. Amin & Khorram [19] analyzed the similar problem with max-product FRE. They also discussed the problem of minimizing a linear objective function subject to the system of max-average FRE. They discussed the feasible solution set with two schemes and studied relationship between maximum and minimum points. They further proposed an algorithm to optimize the objective function. Thapar et al. [54] considered a linear optimization model subject to max-*t* FRE, *t* being an Archimedean t-norm. They applied the concept of covering problem to establish a 0-1 integer programming problem equivalent to linear programming problem and proposed a binary coded genetic algorithm to obtain the optimal solution. Ghodosian et al. [13] investigated a linear programming problem subject to max-harmonic FRE. They first proved necessary and sufficient conditions to determine the feasibility of the problem and on the behalf of these conditions, an algorithm was presented to find the optimal solution of the problem.

Non-linear optimization model subject to max-min FRE was first considered by Lu and Fang [35]. They proposed a genetic algorithm, started with a randomly generated population and progress to improve solutions by using genetic operators such as crossover and mutation. Wu [62] considered the geometric function with single-term exponent problem subject to max-min FRE. They proved some properties for the problem under the assumption of both negative and non-negative exponents in the objective function and then presented an efficient procedure to solve this optimization problem without looking for all the potential minimal solutions. Khorram & Hassanzadeh [20] considered the same model subject to max-average FRE. Since the solution set of such problem is a non-convex set, so the conventional non-linear programming methods are not ideal for solving such a problem. They proposed a genetic algorithm to solve the optimization problem.

Lin et al. [31] proposed a genetic algorithm to optimize a non-linear objective function subject to max-*t* FRE, *t* being a triangular norm. They presented results for max-min and max-product composition. The proposed algorithm first locates the feasible domain through the maximum solution and the

minimal solutions of FRE, to significantly reduce the search space. Zhou et al. [74] considered a posynomial geometric programming function subject to max-min FRE. They converted the posynomial geometric programming function into an equivalent programming problem whose objective function is a non-decreasing function. Furthermore, they presented some rules for simplifying the problem and proposed an algorithm to obtain the optimal solution using the concept of path.

Loetamonphong et al. [34] presented a genetic algorithm to solve a multi-objective optimization problem subject to max-min FRE. They proposed a stochastic searching method which explores the solution space by evaluating the population at hand and evolving the current population to a new one. Wu & Guu [63] presented a two-phase approach to generate an efficient solution for solving multi-objective optimization problem constrained with max-*t* FRE and designed a parameter to connect the min operator and the average operator models so that other efficient solutions can be generated.

A multi-objective optimization model subject to the system of max-average FRE was considered by Khorram & Zarei [22]. They modified the genetic algorithm proposed by Loetamonphong et al. [34] to solve the problem. Thapar et al. [55] presented a genetic algorithm to find Pareto optimal solutions for solving multi-objective problem subject to max-product FRE observing the role of non-convexity of the feasible domain of FRE. Khorram et al. [21] presented some theorems and results to simplify linear fractional multi-objective optimization problem subject to max-Archimedean FRE. They first reduced the feasible region of the optimization problem then converted the linear fractional multi-objective optimization problem to a linear one.

Bipolarity exists widely in human understanding of information and preference [9]. Bipolar representation seems to be useful in the development of intelligence technologies. In [5], the authors tried to show how the possibility theory framework is convenient for handling bipolar representations. There are three forms of bipolarity: symmetric univariate, dual bivariate (or symmetric bivariate unipolarity), and asymmetric (or heterogeneous) bipolarity [9].

Freson et al. [11] proposed the concept of bipolar FRE for the first time, with an application in public awareness of the products for a supplier. They considered an optimization problem subject to max-min bipolar FRE. To solve such problem, they transformed the objective function into an increasing

function and decomposed each bipolar constraint into two systems of unipolar max-min constraint. Li and Liu [28] discussed the same problem with max-Łukasiewicz bipolar FRE. They reformulated the problem into a 0-1 integer linear optimization problem and then solved it by using well developed techniques of integer optimization. They also showed that the determination of the consistency of bipolar max-Łukasiewicz FRE is *NP*-complete and each component of the optimal solution obtained from such an optimization problem can either be the corresponding component lower bound or upper bound value. Due to this characteristic Aliannezhadi et al. [1] analyzed the similar problem with max-parametric hamacher composition. To solve such problem, they presented necessary and sufficient conditions for checking the consistency of the system and proposed some rules for reducing the dimensions of the problem and assigned values to some optimal variables without solving the problem. Aliannezhadi and Molai [2] proposed some sufficient conditions to reduce the size of the search domain of the max-product bipolar FRE and with the help of these conditions they designed algorithm to solve the problem. Moreover, they proposed a modified branch-and-bound method based on a value matrix to solve the reduced problem.

Li and Jin [29] investigated bipolar FRE with max-min composition and showed that the problem of determining their consistency is *NP*-complete. Yang [70] considered max-Łukasiewicz bipolar FRE and proposed a path approach to solve the system and found that the complete solution set of the bipolar system is fully determined by its conservative bipolar paths. Tiwari and Thapar [56] used the idea of covering and the concept of leading, non-leading variables to solve the max-Archimedean bipolar FRE. They showed that the problem of finding the complete solution set of the system of max-Archimedean bipolar FRE is equivalent to solving a covering problem and the solutions of such equations correspond to irredundant coverings of the covering problem. Hamidirad [17] discussed the consistency of bipolar max-T_D FRE, where T_D is the drastic negation operator. They presented special characterization of the feasible domain, maximal and minimal solutions for such problem. Furthermore, some necessary conditions and sufficient conditions were proposed for the solvability of the system of bipolar max-T_D FRE.

Zhou et al. [73] considered the non-linear optimization problem with max-Łukasiewicz bipolar FRE. They reformulated the system of equations into a system of 0-1 mixed integer inequalities and

obtained the optimal solution of the problem. Aliannezhadi et al. [4] considered maximizing a monomial geometric objective function with max-product bipolar FRE. They showed that the bipolar max-product FRI system can equivalently be converted to a bipolar max-product FRE system. They also derived sufficient conditions for solution existence of its feasible domain and designed an algorithm to solve the optimization problem with regard to the structure of its feasible domain and converted the objective function to an increasing function. Mazraeh & Molai [39] considered non-linear optimization problem subject to max-min bipolar FRE. They proposed genetic algorithm which uses the structure of the feasible domain of the problem and lower and upper bounds of the components to choose the initial population. Aliannezhadi & Molai [3] analyzed the geometric programming problem with max-product bipolar FRE. To solve such problem, they discussed over the lower and upper bound of the solution set, and proposed some rules to determine its optimal components without its resolution and on the behalf of these conditions and results; an efficient algorithm was developed to solve the problem using simplification operations.

Hu et al. [18] considered multi-objective optimization problem with max-product bipolar FRE. An integer optimization based technique was proposed for solving such a problem. Some critical features associated with the feasible domain and the optimal solution was studied by them.

Fuzzy relation inequalities (FRI) are important elements of fuzzy mathematics, and they have been widely applied in the fuzzy comprehensive evaluation and cybernetics. Bit-Torrent like P2P file sharing system can be formulated in terms of special type of FRE known as FRI [25,72]. Bit-Torrent (BT) protocol is use in peer-to-peer (P2P) file transfer protocol. Wang et al. [59] studied the properties of max-min FRI, based on which an effective algorithm was developed to deal with the corresponding fuzzy relation latticized optimization problem. They proposed conservative path method to find out all the minimal solutions.

Guo & Xia [16] considered linear optimization function with max-min FRI. They first defined necessary condition for a minimum solution being an optimal solution and then proposed an algorithm to obtain the optimal solution by deleting the quasi-minimum solutions not satisfying the necessary condition of the optimality. Ghodousian & Khorram [12] studied a linear optimization problem subject to max-min FRI in which fuzzy inequalities replace ordinary inequalities in the constraints. Fuzzy

inequalities enable to attain the optimal points that are better solutions than those resulting from the resolution of the similar problems with ordinary inequality constraints. They presented an algorithm to generate such optimal solutions.

Molai [40] worked on quadratic function with max-product FRI. Some sufficient conditions were proposed to determine the optimal solution of the problem and some rules were given to reduce the size of the problem by finding some components of the optimal solutions. Finally, an algorithm was suggested to solve the original problem by combining the sufficient conditions, simplification procedures and the modified simplex method or the numerical algorithms. Yang [68] developed a pseudo-minimal-index algorithm to minimize a linear objective function with addition-min FRI as constraints. Firstly, the nature of the minimal solutions of the inequalities was analyzed, the definitions of the pseudo-minimal indexes and minimal intervals of this system were given and finally an algorithm was proposed for finding pseudo-minimal indexes. With the help of these indexes, they obtained the optimal solution of the problem. To improve the results presented in [68]. Yang et al. [72] considered addition-min FRI and introduced the concept of lexicographic minimum solution and found that the unique lexicographic minimum solution can be selected from the set of minimal solutions of FRI. The minimal solution set was probably infinite, therefore due to this difficulty, they proposed circulation algorithm to find the unique lexicographic minimum solution.

Zhou et al. [75] considered min-product FRI to establish the model for the pricing relation in a supply chain system. They presented the basic properties of the min-product FRI, then characterized the complete solution set with the help of these properties and investigated lexicographic maximum solution as an optimal price scheduling that maximizes the benefits. They proposed an algorithm to search the unique lexicographic maximum solution. Lin et al. [30] also worked on the similar problem with min-product composition. They first studied the resolution of min-product FRI system. They further investigated a maxi-min programming problem subject to the min-product FRI and proposed an algorithm to obtain the optimal solution based on the quasi-maximal matrix and the corresponding index set.

When an arbitrary line fault appears between the emission basic station and the signal testing point in P2P, the classical max-product FRI system is no longer effective for considering an arbitrary line fault

in the wireless communication station system. Therefore, Qiu and Yang [46] introduced and investigated the so-called arbitrary-term-absent (ATA) max-product FRI to reduce the damage caused by the electromagnetic radiation from the basic stations. Since the structure of solution set to the ATA max-product FRI shows that the minimal solutions are not unique in most cases hence, they further introduced the lexicographic order and defined the relevant concept of lexicographic minimal solution. An efficient algorithm with polynomial complexity was developed to obtain the unique lexicographic minimal solution of the ATA max-product system. Some of the relevant properties of addition-min FRI were investigated by Yang [71] for reducing the network congestion under a fixed priority grade of the terminals. With the help of these properties, they developed an efficient algorithm to find the unique lexicographic minimum solution.

Qiu and Yang [47] considered FRI with addition-Łukasiewicz-product composition which have an application in urban sewage treatment systems. Due to this mix composition the solution set was much different. So, they first analyzed the feasible region of the system and then presented some properties for solving the system.

Recently, some of the researchers derived methods to obtaining minimal solutions of FRI. Matusiewicz and Drewniak [38] considered max-product FRI with the increasing operation and investigated some useful properties and developed a novel algorithm to find the set of all minimal solutions. Wu et al. [66] study the properties of the minimal solutions of addition-min FRI and proposed an iterative algorithm to find the minimal solutions. The proposed algorithm was not only able to find the minimal solutions efficiently but also able to compute many minimal solutions in different iterative sequences of variables.

Interval-valued fuzzy set theory is an extension of fuzzy theory in which a closed subinterval of the unit interval is assigned as the membership degree. Therefore, it is very significant to study interval-valued FRE from both the theoretical and practical viewpoints. Wang and Chang [58] investigated the properties and the solution procedure of composite max-min interval-valued FRE. They developed a generalized method to solve an interval-valued FRE and investigated the structure of the solution set and showed that all the solutions lie between the value of the largest one and any smallest one. Li and Fang [24] converted the max-min interval-valued FRE into the system of FRI and proposed an

efficient computational procedure to generate the whole solution set by finding the maximum solution first and then the minimal ones.

Wang et al. [61] introduced three types of solution sets for interval-valued FRE with a min-*s*-norm composition, *s* being a triangular co-norm. It was proved that each of the three solution sets can be found by solving a corresponding FRI system and showed that when the solution set is not empty, it can be represented by a finite set of interval vectors with the minimum solution as the lower bound and one of the maximal solutions as an upper bound. To correct the errors in Wang et al. [60], Li and Fang [27] represented their major results succinctly and correctly to illustrate the relations between interval-valued max-T equations and max-T inequalities, T being a triangular norm, and noticed that similar errors have been found in Wang et al. [61]. Tiwari and Thapar [57] studied the max-Archimedean interval-valued FRE. They analyzed the structure and the properties of the tolerable solution set and proposed an efficient method based on the concept of covering which computes all minimal solutions and concluded that concept of covering is useful for large size problems.

In the above description we discussed distinct methods to find the solution for different types of FRE with different compositions. But in the real world when the problem is formulated in terms of FRE, then it is not necessary that it is consistent i.e. the solution of FRE may not exist. But due realistic nature we need to find the solution for the problem, known as an approximate solution. Various authors have proposed alternative methods to come up with approximate solutions. Pedrycz [45] proposed an approximate solution for the max-min FRE. He tried to find the approximate solution of inconsistent FRE $x \circ A = b$ such that it has minimum distance from b. Unfortunately, the structure of the approximate solution set obtained by this method was not clear. Gottwald and Pedrycz [15] proposed methods based on the equality index introduced for fuzzy sets by Gottwald [14]. The larger value of the solvability index makes easier to solve the equation. In this method, the given equation is appropriately modified to increase the solvability index until an equation is obtained that is solvable. But this method was rather inefficient and leads usually to a trivial solution.

Klir and Yuan [24] generalized approximate solutions of sup-*t* FRE, *t* being any continuous norm, by employing the equality index proposed by Gottwald [14]. They introduced a goodness measure of the performance of approximate solutions and derived a lower bound and an upper bound of solvability of

the system of FRE. Yang [69] established an evaluation model by the satisfaction degree, for comparing two arbitrary potential solutions of max-min FRI. Based on the evaluation model, the concept of approximate solution was defined. It is indeed a potential solution with highest satisfaction degree. Furthermore, effective algorithm was developed for obtaining the approximate solutions for inconsistent system. Xiao et al. [67] introduced a new definition of approximate solution for an inconsistent system of max-min FRE. They first presented some rules for approximate solution and then developed a linear search algorithm based on these rules for obtaining the approximate solutions.

5. PROPOSED WORK AND METHODOLOGY

Many researchers have worked on FRE and FRE constraining linear, non-linear, and multi-objective functions. As we have seen in the literature that the methods and feasible region of FRE vary with the types of FRE. Further, the necessary and sufficient conditions of FRE change when we change the composition of FRE. Thus the resolution of FRE is one of the complex problems of fuzzy set theory.

The study of FRE and FRE associated with different types of objective functions provides the mathematical formulation of practical applications of real world problems. For instance, FRI plays an important role in solving P2P file sharing problem, bi-polar FRE have different applications in the field of economics and bit-torrent. Since, the solution set of such types of problems is non-convex, hence we are unable to apply the traditional techniques, used for solving FRE.

Here, we have mentioned some research areas, we intend to consider:

- The study of different types of FRE and FRI;
- Establishing necessary and sufficient conditions of their solvability and resolution;
- Finding efficient algorithms with less computational complexity to obtain the complete solution set;
- The study of optimization problems (linear, nonlinear, multi objective) constrained with FRE or FRI;
- Proposing efficient algorithms and techniques to solve optimization problems constrained with FRE or FRI for obtaining optimal or converging solutions;

- The development of techniques to find the approximate solutions for inconsistent system of FRE or FRI;
- Applications of FRE and FRI in real life instances.

The methodology to be adopted for the proposed work includes fuzzy relational calculus and its meta mathematics, generalized classical optimization techniques, matrix algorithms, computer based algorithms, hybrid algorithms and evolutionary algorithms.

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